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Quantum Number Theory

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This paper shows that a theory of numbers can be developed within the framework of typed quantum logic. First, the core system of typed quantum logic is stated precisely, on which a theory of numbers is to be built. Second, under a minimal additional assumption about natural number objects, a step-by-step description of quantum real numbers is provided. Finally, it is suggested that the "quantum reals" are modeled by the "observables" in terms of quantum physics.

KEY WORDS: Quantum logic; type theory; theory of numbers.

1. INTRODUCTION

Based on typed quantum logic, this paper presents a formal theory of numbers to describe the general area of quantum mathematics. In the previous paper (Tokuo, 2003), we have proposed a basic system of typed quantum logic. We restate its syntax and semantics here in a form suitable to the present context, and add new axioms in order to incorporate the concept of numbers. Specifically, we need the axiom of infinity to ensure that there exist an infinite number of individuals that are distinct from each other. The most common representation of the axiom is known as the *Peano rules*:

$$Sn = 0 \vdash p \tag{1.1}$$

$$Sm = Sn \vdash m = n \tag{1.2}$$

$$\frac{\vdash \phi(0)\ \phi(m) \vdash \phi(Sm)}{\vdash \phi(n)} \tag{1.3}$$

where m, n is a term of type N (i.e., numerals); p is a term of type Ω (i.e., a formula); 0 is the unique constant of type N; S is a term-forming operator which acts on a term of type N, and produces another term of type N.

We denote the set of natural numbers by $\mathbf{N}$. To be precise, $\mathbf{N} \equiv \{x \in N | \top\}$. In the context of classical logic, the Peano rules and the following *Simple Recursion*

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Principle (SRP) are provably equivalent.

$$a \in \alpha, h \in \alpha^{\alpha} \vdash \exists! f \in \alpha^{\mathbf{N}}.((f(0) = a) \wedge (f(Sn) = h(f(n))) \qquad (1.4)$$

where α is a given set; α^{α} ($\alpha^{\mathbf{N}}$) is a set of all functions from α to α (from **N** to α), respectively. Intuitively, SRP asserts that for a given set α, a given function h on α and a given element a of α, there exists a unique sequence of α that is recursively generated. Thus, classically, it does not matter which of the rules are chosen as axioms (for proofs, see Bell, 1988.)

In quantum logic, however, the situation is more complicated. Since quantum logic is weaker than classical logic in the sense that the former has fewer theorems (derivable propositions) than the letter, we need to be careful in choosing a minimal set of axioms to derive desirable theorems. In view of this, we postulate SRP as one of the axioms, and consider the Peano rules and other principles of number theory as derivable ones. What needs to be emphasized throughout this paper is that such derivations can be carried out formally.

2. FORMAL SYNTAX

The language $\mathcal{L}$ for typed quantum logic being introduced here is essentially a corrected version of that given in Tokuo (2003), the predecessor of this paper. To prove the theorems listed in Section 4 of the paper, we have had to assume (a weak form of) the rule of extensionality (2.19).

2.1. Terms

A term is a meaningful expression in our formal language. Each term has a property called *type*.

Definition 2.1. (Type).

(i) Ω is a special type. (Truth type)
(ii) At most countable symbols $C_1, C_2, C_3, \ldots$ denote general types. (Atomic type)
(iii) If A is a general type, so is PA. (Power type)
(iv) If A and B are general types, so is $A \times B$. (Product type)
(v) Nothing else is a type except as defined by (i) and a finite number of applications of (ii)–(iv).

A term of type Ω is called a *formula*. For any general type A, a term of type PA is called a *set-like term*. A term with no free variables is called *closed*. A closed set-like term and a closed formula are called an $\mathcal{L}$*-set* and a *sentence*, respectively.

Definition 2.2. (Term).

(i) For each type A, countably many symbols $x_A, y_A, z_A, \ldots$ denote variables of type A (type subscripts are mostly omitted for simplicity). A variable of each type is a term of that type.
(ii) For each type A, countably many symbols $c_A, d_A, e_A, \ldots$ denote constants of type A (type subscripts are mostly omitted for simplicity). A constant of each type is a term of that type.[2]
(iii) If $\phi(x)$ is a formula that possibly contains a variable x of general type A, then $\{x \in A|\phi(x)\}$ is a set-like term of type PA.
(iv) If a and b are terms of general type A and B, respectively, then $\langle a, b\rangle$ is a term of type $A \times B$.
(v) If a is a term of type $A \times B$, then $(a)_1$ and $(a)_2$ are terms of type A and type B, respectively.
(vi) If a and a' are terms of the same type, then $a = a'$ is a formula.
(vii) If a and α are terms of general type A and type PA, respectively, then $a \in \alpha$ is a formula.
(viii) If p and q are formulas, then so is $p \wedge q$.
(ix) if p is a formula, then so is $\neg p$.
(x) If $\phi(x)$ is a formula that possibly contains a variable x of type A, then $\forall x \in A.\phi(x)$ is a formula.
(xi) Nothing else is a term except as defined by a finite number of applications of the above clauses.

Notation. Parentheses are used to disambiguate expressions as usual. Some other useful symbols can be introduced as abbreviations:

- $p \vee q \equiv \neg(\neg p \wedge \neg q)$
- $p \Rightarrow q \equiv \neg p \vee (p \wedge q)$
- $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\exists x \in A.\phi(x) \equiv \neg(\forall x \in A.\neg\phi(x))$
- $\exists! x \in A.\phi(x) \equiv \exists x \in A.(\phi(x) \wedge \forall y \in A.(\phi(y) \Rightarrow (x = y)))$
- $\top \equiv \neg(p \wedge \neg p)$
- $\bot \equiv \neg\top$
- $\{\langle x, y\rangle \in A \times B|\phi(x, y)\} \equiv \{z \in A \times B|\exists x \in A.(\exists y \in B.(z = \langle x, y\rangle \wedge \phi(z)))\}$
- $\forall x \in \alpha.\phi(x) \equiv \forall x \in A.((x \in \alpha) \Rightarrow \phi(x))$
- $\exists x \in \alpha.\phi(x) \equiv \exists x \in A.((x \in \alpha) \wedge \phi(x))$
- $\exists! x \in \alpha.\phi(x) \equiv \exists! x \in A.((x \in \alpha) \wedge \phi(x))$
- $\{x \in \alpha|\phi(x)\} \equiv \{x \in A|(x \in \alpha) \wedge \phi(x)\}$

[2] The assumption that there are an infinite number of constants is not essential but it simplifies the arguments in Theorem 3.6.

The usual set-theoretic operations and relations are defined as follows:

- $\{a\} \equiv \{x \in A | x = a\}$
- $\alpha \subseteq \beta \equiv \forall x \in \alpha . x \in \beta$ (where α and β are of the same type PA.)
- $\alpha \cap \beta \equiv \{x \in A | (x \in \alpha) \wedge (x \in \beta)\}$ (where α and β are of the same type PA.)
- $\alpha \cup \beta \equiv \{x \in A | (x \in \alpha) \vee (x \in \beta)\}$ (where α and β are of the same type PA.)
- $\cap U \equiv \{x \in A | \forall y \in U . x \in y\}$
- $\cup U \equiv \{x \in A | \exists y \in U . x \in y\}$
- U_A or $A \equiv \{x \in A | \top\}$
- ϕ_A or $\phi \equiv \{x \in A | \bot\}$
- $-\alpha \equiv \{x \in A | \neg(x \in \alpha)\}$
- $P\alpha \equiv \{x \in PA | x \subseteq \alpha\}$
- $\alpha \times \beta \equiv \{\langle x, y\rangle \in A \times B | (x \in \alpha) \wedge (y \in \beta)\}$ (where α is of type PA and β is of type PB. Both may be of the same type.)
- $\beta^{\alpha} \equiv \{x \in P(A \times B) | (x \subseteq (\alpha \times \beta)) \wedge \forall y \in \alpha . (\exists ! z \in \beta . \langle y, z\rangle \in x)\}$ (where α is of type PA and β is of type PB. Both may be of the same type.)
- $gf \equiv \{\langle x, z\rangle \in \alpha \times \gamma | \exists y \in \beta . ((\langle x, y\rangle \in f) \wedge (\langle y, z\rangle \in g))\}$

2.2. Rules

We now state the formal proof procedure for $\mathcal{L}$. In the following, we write $A, B, \ldots$ for types, $p, q, \ldots$ for formulas, $a, b, \ldots$ for terms, and $x, y, \ldots$ for variables. $\phi(x)$ represents a formula that possibly contains a free variable x, $\phi(a)$ a formula obtained from $\phi(x)$ by replacing all free occurrences of x with a. The substitution is performed in the usual manner, that is, variables are renamed to avoid free variable capture if necessary. For notational simplicity, we exclusively consider formulas with at most one free variable; the modification for multiple free variables is easy to perform. Γ denotes a finite (possibly empty) multiset of formulas, where a multiset means a set in which each element may occur more than once.

The intuitive meanings of the clauses listed below are as follows. The expression of the form $\Gamma \vdash p$ is called a *sequent* (Bell, 1988), asserting that one can syntactically deduce the formula p from the assumption of all the formulas in Γ. The expressions $\Gamma, p \vdash q$ and $p, q \vdash r$ mean $\Gamma \cup \{p\} \vdash q$ and $\{p, q\} \vdash r$, respectively. The sequents above the horizontal line are the premises of the rule, and the one below it is the conclusion of the rule: if all the sequents above the line hold, so does the one below it. The one-line rules such as (2.1) below are sometimes called *improper rules*, asserting that those sequents always hold without any premises.

Definition 2.3. (Formal Proof). A diagram of rules that satisfies the following inductive specifications is said to be a *proof diagram*. The bottommost sequent of a proof diagram is called its *end sequent*.

(i) An improper rule is itself a proof diagram.
(ii) If P is a proof diagram whose end sequent is S, and S/T is one of the rules listed above, then P/T is a proof diagram whose end sequent is T.[3]
(iii) If P_1 and P_2 are both proof diagrams whose end sequents are S_1 and S_2, respectively, and $S_1\ S_2/T$ is one of the rules listed above, then $P_1\ P_2/T$ is a proof diagram whose end sequent is T.

A proof diagram whose end sequent is T is called a proof diagram of T. We say that the sequent T is *provable* if and only if there exists a proof diagram of T. We often omit the phrase "is provable" and simply write "$\Gamma \vdash p$" to mean the sequent is provable.

Structural rules:

$$p \vdash p \tag{2.1}$$

$$\frac{\Gamma \vdash p \quad \Gamma, p \vdash q}{\Gamma \vdash q} \qquad \text{(Cut)} \tag{2.2}$$

$$\frac{\Gamma \vdash q}{\Gamma, p \vdash q} \tag{2.3}$$

$$\frac{\Gamma(x) \vdash \phi(x)}{\Gamma(a) \vdash \phi(a)} \tag{2.4}$$

where any free variables occurring in a are not bounded in the lower sequent.

Logical rules:

$$p \wedge q \vdash p \tag{2.5}$$

$$p \wedge q \vdash q \tag{2.6}$$

$$\frac{\Gamma \vdash p \quad \Gamma \vdash q}{\Gamma \vdash p \wedge q} \tag{2.7}$$

$$p \vdash \neg\neg p \tag{2.8}$$

$$\neg\neg p \vdash p \tag{2.9}$$

$$p \wedge \neg p \vdash q \tag{2.10}$$

$$\frac{p \vdash q}{\neg q \vdash \neg p} \tag{2.11}$$

[3] We have written a slash ('/') for a horizontal line.

$$\forall x \in A.\phi(x) \vdash \phi(a) \tag{2.12}$$

where variables are renamed if necessary to avoid free variable capture.

$$\frac{\Gamma \vdash \phi(x)}{\Gamma \vdash \forall x \in A.\phi(x)} \tag{2.13}$$

where x does not occur freely in Γ.

$$p, p \Rightarrow q \vdash q \quad \text{(Orthomodular law)} \tag{2.14}$$

Equality rules:

$$\vdash a = a \tag{2.15}$$

$$a \; = a', \phi(a) \vdash \phi(a') \tag{2.16}$$

where variables are renamed if necessary to avoid free variable capture.

$$\frac{p \vdash q \quad q \vdash p}{\vdash p = q} \tag{2.17}$$

Set rules:

$$\vdash (a \in \{x \in A | \phi(x)\}) = \phi(a) \quad \text{(Comprehension)} \tag{2.18}$$

$$(x \in \alpha) = (x \in \beta) \vdash \alpha = \beta \quad \text{(Extensionality)} \tag{2.19}$$

$$\vdash ((\langle a, b \rangle)_1 = a) \wedge ((\langle a, b \rangle)_2 = b) \tag{2.20}$$

$$\vdash \langle (a)_1, (a)_2 \rangle = a \tag{2.21}$$

Remark. $\vdash f \in \beta^\alpha$ is a sequent asserting that f is a term representing a function from α to β, where α and β are set-like terms. In such cases, it is often desirable to introduce an explicit term representing the function value $f(x)$ such that $x \in \alpha \vdash \langle x, f(x) \rangle \in f$.

Strictly, this convention is justified by the fact that adding a new term $f^*(x)$ and a new rule $x \in \alpha \vdash \langle x, f^*(x) \rangle \in f$ to $\mathcal{L}$ yields a *conservative extension* of $\mathcal{L}$. For technical details, the reader is referred to Bell (1988). Bearing this in mind we simply write $f(x)$ for $f^*(x)$ in this paper.

In the same spirit, if $\vdash \exists x \in A.\phi(x)$ holds, then one may temporarily use a new term c of type A such that $\phi(c)$.

3. FORMAL SEMANTICS

This section provides mathematically strict meanings of the expressions in $\mathcal{L}$. For this purpose, a class of models for $\mathcal{L}$ is specified. First, the precise concept of validity is defined with respect to this class of models. Some important results are stated: any provable sequents are valid with respect to this class of models (*soundness*), and conversely, any valid sequents with respect to this class of models are provable (*completeness*). To prove a completeness theorem, a concrete general model, called the *canonical model*, is constructed.

3.1. General Model

Definition 3.1 (Frame). For each type A, its *domain* D_A is defined as a nonempty set satisfying the following conditions.

- For the truth type Ω, its domain D_Ω is a nonempty orthomodular lattice. Its order, equality, inf, orthocomplementation, top and bottom are denoted by $\leq$, $=$, $\wedge$, $*$, $\top$ and $\perp$, respectively.[4]
- For each power type PA, its domain D_{PA} is a nonempty subset of $D_\Omega^{D_A}$ (the set of all functions from D_A to D_Ω).
- For each product type $A \times B$, its domain $D_{A\times B}$ is the product set $D_A \times D_B$.

A collection $\{D_A\}_A$ of domains of all types is called an $\mathcal{L}$-*frame*.

Definition 3.2 (Assignment). Given an $\mathcal{L}$-frame $\{D_A\}_A$, an *assignment* ρ is defined as a map on the sets of all variables in $\mathcal{L}$, satisfying the condition that $\rho(x_A) \in D_A$ for each type A. Given an assignment ρ, a variable x_A and an element $\delta \in D_A$, we write $(\rho : x_A/\delta)$ for the assignment which is the same map as ρ except that the value of x_A is δ.

Definition 3.3 (General Model). An $\mathcal{L}$-frame $\{D_A\}_A$ is said to be a *general model* $\mathcal{M}$ for $\mathcal{L}$ if there exists a map $[\cdot]_\rho$ from the set of all terms to their domains, satisfying the following conditions.

(i) $[x_A]_\rho = \rho(x_A)$
(ii) $[c_A]_\rho \in D_A$
(iii) $[\{x \in A|\phi(x)\}]_\rho = ([a]_\rho \mapsto [\phi(a)]_\rho)$
(iv) $[(x \in \alpha) = (x \in \beta)]_\rho \leq [\alpha = \beta]_\rho$
(v) $[\langle a, b\rangle]_\rho = ([a]_\rho, [b]_\rho)$
(vi) $[(\langle a, b\rangle)_1]_\rho = [a]_\rho; [(\langle a, b\rangle)_2]_\rho = [b]_\rho$
(vii) $[a = a']_\rho = \top$ if $[a]_\rho = [a']_\rho$
(viii) $[a = a']_\rho \wedge [\phi(a)]_\rho \leq [\phi(a')]_\rho$
(ix) $[a \in \alpha]_\rho = [\alpha]_\rho([a]_\rho)$
(x) $[p \wedge q]_\rho = [p]_\rho \wedge [q]_\rho$
(xi) $[\neg p]_\rho = [p]_\rho^*$
(xii) $[\forall x \in A.\phi(x)]_\rho = \bigwedge_{\delta\in D_A}\{[\phi(x)]_{(\rho:x/\delta)}\}$[5]

[4] Some symbols are the same as our logical symbols; this is not expected to cause confusion since it is clear from the context.
[5] This condition implies that the limit on the right side exists and is equal to the left side.

Definition 3.4 (Validity). Let $\mathcal{M}$ be a general model for $\mathcal{L}$. For $\Gamma \equiv \{p_1, p_2, \ldots, p_m\}$, we define $[\Gamma]_\rho$ as $[\rho_1]_\rho \wedge [p_2]_\rho \wedge \cdots \wedge [p_m]_\rho$ if $m > 0$; $\top$ otherwise. We say that $\Gamma \vdash p$ is *valid* in $\mathcal{M}$, or symbolically we write $\Gamma \models_{\mathcal{M}} p$, if $[\Gamma]_\rho \leq [p]_\rho$ for any ρ. We write $\Gamma \models p$ if $\Gamma \models_{\mathcal{M}} p$ in every general model $\mathcal{M}$.

3.2. Soundness and Completeness

Theorem 3.5. *(Soundness). If $\Gamma \vdash p$, then $\Gamma \models p$.*

Proof: We can show that the end sequent in every proof diagram is valid in every general model, by induction on the construction of the proof. Given any model, the improper rule is obviously valid. For the induction step, one may routinely verify that for each rule, if all the premises are valid, then the conclusion is also valid. □

Theorem 3.6. *(Completeness). If $\Gamma \models p$, then $\Gamma \vdash p$.*

Proof: The proof begins by introducing the Lindenbaum algebra of $\mathcal{L}$. Let us define a relation $\sim$ on the set of all terms: $a \sim a'$ if and only if the sequent $\vdash a = a'$ is provable. It is easy to see that this relation is an equivalence relation: a relation that is reflexive, symmetric, and transitive. The equivalence class of a is denoted by $[a]$. For each type A, let D_A be the set of all the equivalence classes of *closed terms* of type A. The set of all equivalence classes of *sentences* forms an orthomodular lattice with the following order relation.

$$[p] \leq [q] \textit{ if and only if } p \vdash q$$

This $\mathcal{L}$-frame $\{D_A\}_A$ with the map $[a]_\rho \stackrel{\text{def}}{=} [a_\rho]$ is called the *canonical model* $\mathcal{M}^{\text{can}}$ for $\mathcal{L}$, where a_ρ means the closed term obtained from a by replacing all free occurrences of x with the closed term c such that $\rho(x) = [c]$. Then we see that $\mathcal{M}^{\text{can}}$ is indeed a general model; the only nontrivial part is to verify (xii) in Definition 3.3. The inequality

$$[\forall x \in A.\phi(x)]_\rho \leq [\phi(c)]_\rho \text{ for any closed term } c \text{ of type } A$$

follows from the axiom (2.12). For the other direction of the inequality, suppose that $[p]_\rho \leq [\phi(c)]_\rho$ for any closed term c of type A. This means that $p \vdash \phi(c_A)$ is provable for any constant c_A. Let c_A be a fresh constant that does not occur in the sequent $p \vdash \phi(x)$. We denote by Π the proof diagram of $p \vdash \phi(c_A)$. Replacing all occurrences of variable x in Π, if any, with a fresh variable, and replacing all occurrences of c_A in Π with x, we obtain the proof diagram of $p \vdash \phi(x)$. Applying the axiom (2.13) yields the proof diagram of $p \vdash \forall x \in A.\phi(x)$, which means that $[p]_\rho \leq [\forall x \in A.\phi(x)]_\rho$. □

The completeness proof using the canonical model $\mathcal{M}^{\text{can}}$ goes as follows. For sequents with no free variables, suppose that $\Gamma \models p$. In particular, $\Gamma \models_{\mathcal{M}^{\text{can}}} p$ for the canonical model $\mathcal{M}^{\text{can}}$. This immediately means that $\Gamma \vdash p$ is provable. For sequents with a free variable x_A, $\Gamma(x_A) \models_{\mathcal{M}^{\text{can}}} \phi(x_A)$ means that $[\Gamma(x_A)]_\rho \leq [\phi(x_A)]_\rho$ for any assignment ρ. Hence we have $[\Gamma(c_A)]_\rho \leq [\phi(c_A)]_\rho$ for any fresh constant c_A, which means that $\Gamma(c_A) \vdash \phi(c_A)$ is provable. We denote by Π the proof diagram of this proof. Replacing all occurrences of variable x in Π, if any, with a fresh variable, and replacing all occurrences of c_A in Π with x, we obtain the proof diagram of $\Gamma(x) \vdash \phi(x)$.

3.3. Consistency

This subsection verifies by constructing a simple concrete model that our typed quantum logic is indeed consistent and does not collapse into classical logic.

- For the truth type Ω, let D_Ω be the orthomodular lattice $L_6 \stackrel{\text{def}}{=} \{a, a^*, b, b^*, \bot, \top\}$ where the only ordering relations are $\bot \leq a \leq \top$, $\bot \leq a^* \leq \top$, $\bot \leq b \leq \top$ and $\bot \leq b^* \leq \top$.
- For each atomic type C, let D_C be some set.
- For each power type PA, let D_{PA} be the set of all functions from D_A to D_Ω.

The interpretation map $[\cdot]_\rho$ is inductively defined as follows.

(i) $[x_A]_\rho \stackrel{\text{def}}{=} \rho(x_A)$.
(ii) Let $[c_A]_\rho$ be some element of D_A.
(iii) $[\{x \in A | \phi(x)\}]_\rho \stackrel{\text{def}}{=} \delta \mapsto [\phi(x)]_{(\rho:x/\delta)}$.
(iv) $[\langle a, b\rangle]_\rho \stackrel{\text{def}}{=} ([a]_\rho, [b]_\rho)$.
(v) $[(\langle a, b\rangle)_1]_\rho \stackrel{\text{def}}{=} [a]_\rho$; $[(\langle a, b\rangle)_2]_\rho \stackrel{\text{def}}{=} [b]_\rho$.
(vi) $[a = a']_\rho \stackrel{\text{def}}{=} \top$ if $[a]_\rho = [a']_\rho$; $\bot$ otherwise.
(vii) $[a \in \alpha]_\rho \stackrel{\text{def}}{=} [\alpha]_\rho([a]_\rho)$.
(viii) $[p \wedge q]_\rho \stackrel{\text{def}}{=} [p]_\rho \wedge [q]_\rho$.
(ix) $[\neg p]_\rho \stackrel{\text{def}}{=} [p]^*_\rho$.
(x) $[\forall x \in A.\phi(x)]_\rho \stackrel{\text{def}}{=} \bigwedge_{\delta \in D_A} \{[\phi(x)]_{\rho(x/\delta)}\}$.

It is easy to see that this frame and interpretation satisfy the conditions in Definition 3.3. Since $\top \neq \bot$ in L_6, the soundness theorem assures the consistency of $\mathcal{L}$.

To see that it is not classical, let c_A, d_A and α_{PA} be constants such that $[\alpha_{PA}]_\rho([c_A]_\rho) \stackrel{\text{def}}{=} a$ and $[\alpha_{PA}]_\rho([d_A]_\rho) \stackrel{\text{def}}{=} b$ for some $[\cdot]_\rho$. Since $a \wedge (b \vee a^*) \not\leq$

$(a \wedge b) \vee (a \wedge a^*)$ in L_6, we have again by the soundness theorem that the sequent $(c_A \in \alpha_{PA}) \wedge ((d_A \in \alpha_{PA}) \vee \neg(c_A \in \alpha_{PA})) \vdash ((c_A \in \alpha_{PA}) \wedge (d_A \in \alpha_{PA})) \vee ((c_A \in \alpha_{PA}) \wedge \neg(c_A \in \alpha_{PA}))$ is unprovable, which means that the distributive law fails in $\mathcal{L}$.

4. NATURAL NUMBERS

In this section, we develop a theory of numbers within the framework of typed quantum logic by adding extra symbols and rules to $\mathcal{L}$.

4.1. Theory of Quantum Numbers

The extended theory $\mathcal{L}^N$ consists of $\mathcal{L}$ together with the following symbols and rules.

Symbols:

- N is a general type. (Number type)
- The special constant 0 is a term of type N.
- S is a term-forming symbol that acts on a term of type N: if n is a term of type N, then so is Sn.

Number rules:

$$\vdash c_N = 0 \tag{4.1}$$

$$a \in \alpha, h \in \alpha^\alpha \vdash \exists! f \in \alpha^{\mathbf{N}}.((f(0) = a) \wedge (f(Sn) = h(f(n))) \quad \text{(SRP)} \tag{4.2}$$

Remark. A term of type N is called a *numeral*. We write $m, n, \ldots$ for numerals, and $\mathbf{N}$ the set of numerals. To be precise, $\mathbf{N} \equiv \{x \in N | \top\}$.

The rule (4.2) is referred to as the *Simple Recursion Principle*, or *SRP*, which guarantees the existence of an infinite sequence.

4.2. Peano Rules

The Peano rules are formulated in $\mathcal{L}^N$ in the following way. What is important is that these rules are *derivable* in $\mathcal{L}^N$.

$$Sn = 0 \vdash p \tag{4.3}$$

$$Sm = Sn \vdash m = n \tag{4.4}$$

$$\frac{\vdash \phi(0) \quad \phi(m) \vdash \phi(Sm)}{\vdash \phi(n)} \tag{4.5}$$

Theorem 4.1. (4.3)–(4.5) *is provable in* $\mathcal{L}^N$.

Proof: We need the following lemma. □

Lemma 4.2. *(Primitive Recursion Principle, PRP).*

$$a \in \alpha, h \in \alpha^{\alpha \times \mathbf{N}} \vdash \exists! f \in \alpha^{\mathbf{N}}.((f(0) = a) \wedge (f(Sn) = h(f(n), n)) \tag{4.6}$$

Proof: Let $\alpha' \equiv \alpha \times N$ and $h' \equiv \{\langle x, y\rangle \in \alpha' \times \alpha' | (x = \langle a, m\rangle) \wedge (y = \langle b, n\rangle) \wedge (b = h(\langle a, m\rangle)) \wedge (n = m)\}$. Applying SRP to α' and h' yields $a' \in \alpha', h' \in \alpha'^{\alpha'} \vdash \exists! f' \in \alpha'^{\mathbf{N}}.(f'(0) = a') \wedge (f'(Sn) = h'(f'(n)))$. We have PRP by letting $f \equiv \{\langle x, y\rangle \in \mathbf{N} \times \alpha | y = (f'(x))_1\}$ and $a \equiv (a')_1$. □

Proof of Theorem: (4.3): Let $a \equiv \bot, \alpha \equiv \{x \in \Omega | (x = \top) \vee (x = \bot)\}$ and $h \equiv \{\langle x, y\rangle \in \alpha \times \alpha | y = \top\}$. Applying SRP to a, α and h yields $\vdash \exists! f \in \alpha^{\mathbf{N}}. ((f(0) = \bot) \wedge (f(Sn) = h(f(n))))$. Hence we have $Sn = 0 \vdash \bot = \top$, that is, $Sn = 0 \vdash p$ for any formula p.

(4.4): Let $a \equiv 0, \alpha \equiv \mathbf{N}$ and $h \equiv \{\langle x, y\rangle \in (\mathbf{N} \times \mathbf{N}) \times \mathbf{N} | y = (x)_2\}$. Applying PRP to a, α and h yields $\vdash \exists! f \in \mathbf{N}^{\mathbf{N}}.((f(0) = 0) \wedge (f(Sn) = n))$. Hence we have $Sm = Sn \vdash f(Sm) = f(Sn)$, that is, $Sm = Sn \vdash m = n$.

(4.5): Suppose $\vdash \phi(0)$ and $\phi(m) \vdash \phi(Sm)$. Let $a \equiv 0, \alpha \equiv \{x \in N | \phi(x)\}$ and $h \equiv \{\langle x, y\rangle \in \alpha \times \alpha | y = Sx\}$. Our aim is to show that $\vdash n \in \alpha$. Applying SRP to a, α and h yields $\vdash \exists! f \in \alpha^{\mathbf{N}}.((f(0) = 0) \wedge (f(Sn) = S(f(n))))$. Obviously, h can be extended to a function h' that is defined on $\mathbf{N}$. Applying SRP to a, $\mathbf{N}$ and h' yields $\vdash \exists! f' \in \mathbf{N}^{\mathbf{N}}.((f'(0) = 0) \wedge (f'(Sn) = S(f'(n))))$. Since we can infer $\vdash f \in \mathbf{N}^{\mathbf{N}}$ from $\vdash f \in \alpha^{\mathbf{N}}$, we obtain $\vdash f = f'$ by the uniqueness condition in SRP. Similarly, letting $\mathrm{id} \equiv \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N} | x = y\}$, we also have $\vdash \mathrm{id} \in \mathbf{N}^{\mathbf{N}}$ and $\vdash (\mathrm{id}(0) = 0) \wedge (\mathrm{id}(Sn) = S(\mathrm{id}(n)))$. Therefore we obtain $\vdash f' = \mathrm{id}$ by the uniqueness condition in SRP. Hence $\vdash f = \mathrm{id}$. This means that $\vdash \langle n, n\rangle \in f$, that is, $\vdash n \in \alpha$. □

4.3. Numerical Functions

Addition of N

Let $a \equiv \mathrm{id}_{\mathbf{N}} \equiv \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N} | y = x\}$ (the identity function on $\mathbf{N}$), $\alpha \equiv \mathbf{N}^{\mathbf{N}}$ and $h \equiv \{\langle x, y\rangle \in \alpha \times \alpha | y = sx\}$, where $s \equiv \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N} | y = Sx\}$. Applying SRP to a, α and h yields the unique function f_+ from $\mathbf{N}$ to α such that

$$\vdash (f_+(0) = \mathrm{id}_{\mathbf{N}}) \wedge (f_+(Sn) = s(f_+(n))). \tag{4.7}$$

We denote $(f_+(m))(n)$ by $m + n$ following the usual convention.

This $+$, as expected, satisfies the usual properties of addition. Henceforth, we use the symbols i, j, k, l, m, n, s, t to refer to numerals. To avoid an execessive use of parentheses, the operator $+$ is assumed to bind stronger than $=$ and the other connectives, e.g., $l = m + n$ is parsed as $l = (m + n)$.

Proposition 4.3. $\vdash 0 + n = n$.

Proof: Immediate from (4.7). □

Proposition 4.4. $\vdash Sm + n = S(m + n)$

Proof: Immediate from (4.7). □

Proposition 4.5. $\vdash n + 0 = n$.

Proof: By formal induction (4.5) on n. □

Proposition 4.6. $\vdash m + Sn = Sm + n$.

Proof: By formal induction (4.5) on m. □

Proposition 4.7. $\vdash m + n = n + m$.

Proof: By formal induction (4.5) on n. For the base case, we have $\vdash m + 0 = 0 + m$ by Proposition 4.3 and 4.5. For the induction step, we must prove

$$m + n = n + m \vdash m + Sn = Sn + m. \tag{4.8}$$

This sequent is rewritten as $m + n = n + m \vdash S(m + n) = S(n + m)$ by Proposition 4.6 and 4.4. Thus (4.8) holds. □

Proposition 4.8. $\vdash (l + m) + n = l + (m + n)$.

Proof: By formal induction (4.5) on n. For the base case, we have $\vdash (l + m) + 0 = l + (m + 0)$ by Proposition 4.5. For the induction step, we must prove

$$(l + m) + n = l + (m + n) \vdash (l + m) + Sn = l + (m + Sn). \tag{4.9}$$

This sequent is rewritten as $(l + m) + n = l + (m + n) \vdash S((l + m) + n) = S(l + (m + n))$ by Proposition 4.6 and 4.4. Thus (4.9) holds. □

Proposition 4.9. $m + n = 0 \vdash (m = 0) \wedge (n = 0)$.

Proof: We prove the equivalent sequent $\vdash (m + n = 0) \Rightarrow ((m = 0) \wedge (n = 0))$ by formal induction (4.5) on n. The base case clearly holds. For the induction step, it is sufficient to prove

$$\vdash (m + Sn = 0) \Rightarrow ((m = 0) \wedge (Sn = 0)). \tag{4.10}$$

This sequent is rewritten as $\vdash \bot \Rightarrow ((m = 0) \wedge (Sn = 0))$ by Proposition 4.6, 4.4 and (4.3). Thus (4.10) holds. □

Proposition 4.10. $l + m = l + n \vdash m = n$.

Proof: We prove the equivalent sequent $\vdash (l + m = l + n) \Rightarrow (m = n)$ by formal induction (4.5) on l. The base case clearly holds. For the induction step, we must prove

$$(l + m = l + n) \Rightarrow (m = n) \vdash (Sl + m = Sl + n) \Rightarrow (m = n). \tag{4.11}$$

This sequent is rewritten as $(l + m = l + n) \Rightarrow (m = n) \vdash (S(l + m) = S(l + n)) \Rightarrow (m = n)$. by Proposition 4.4. Since $S(l + m) = S(l + n) \vdash l + m = l + n$ holds by (4.4), and clearly $l + m = l + n \vdash S(l + m) = S(l + n)$ holds, we obtain $\vdash (S(l + m) = S(l + n)) = (l + m = l + n)$. Thus (4.11) holds. □

Multiplication of $\mathbf{N}$

Let $a \equiv 0_{\mathbf{N}} \equiv \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N} | y = 0\}$ (the zero function on $\mathbf{N}$), $\alpha \equiv \mathbf{N}^{\mathbf{N}}$ and $h \equiv \{\langle x, y\rangle \in \alpha \times \alpha | y = x + \mathrm{id}_{\mathbf{N}}\}$, where addition of functions is defined pointwise in the obvious way. Applying SRP to a, α, and h yields the unique function $f_{\times}$ from $\mathbf{N}$ to α such that

$$\vdash (f_{\times}(0) = 0_{\mathbf{N}}) \wedge (f_{\times}(Sn) = (f_{\times}(n) + \mathrm{id}_{\mathbf{N}})). \tag{4.12}$$

We denote $(f_{\times}(m))(n)$ by $m \cdot n$, or simply mn, following the usual convention. To avoid an excessive use of parentheses, the operator $\cdot$ is assumed to bind stronger than $+$, $=$ and the other connectives.

For later use, we list some propositions on multiplication.

Proposition 4.11. $\vdash 0n = 0$.

Proof: Immediate from (4.12). □

Proposition 4.12. $\vdash (Sm)n = mn + n$.

Proof: Immediate from (4.12). □

Proposition 4.13. $\vdash n(S0) = n$.

Proof: By formal induction (4.5) on n. □

Proposition 4.14. $\vdash l(m+n) = lm + ln$.

Proof: By formal induction (4.5) on l. □

Proposition 4.15. $\vdash mn = nm$.

Proof: By formal induction (4.5) on m, using Proposition 4.13 and 4.14. □

Proposition 4.16. $\vdash (lm)n = l(mn)$.

Proof: By formal induction (4.5) on l, using Proposition 4.14 and 4.15. This proposition allows us to omit parentheses in expressions such as $(lm)n$ and $l(mn)$. □

Order on ***N***

Now we proceed to define order on $\mathbf{N}$. We set $O_{\mathbf{N}} \equiv \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N} \mid \exists z \in \mathbf{N}.(y = x + z)\}$ and write $m \le n$ for $\langle m, n\rangle \in O_{\mathbf{N}}$. To see that this relation $\le$ is indeed a linear order relation, we need to check if it satisfies reflexivity, transitivity, antisymmetry, and linearity.

Proposition 4.17. *(Reflexivity).* $\vdash n \le n$.

Proof: Immediate from Proposition 4.5. □

Proposition 4.18. *(Transitivity).* $l \le m, m \le n \vdash l \le n$.

Proof: Immediate from Proposition 4.8. □

Proposition 4.19. *(Antisymmetry).* $m \le n, n \le m \vdash m = n$.

Proof: Immediate from Proposition 4.9 and 4.10. □

Proposition 4.20. *(Linearity).* $\vdash (m \le n) \vee (n \le m)$.

Proof: By formal induction (4.5) on m. The base case is clear since $\vdash 0 \le n$. For the induction step, we must prove $(m \le n) \vee (n \le m) \vdash (Sm \le n) \vee (n \le Sm)$, that is, we must prove

$$m \le n \vdash (Sm \le n) \vee (n \le Sm) \tag{4.13}$$

and

$$n \leq m \vdash (Sm \leq n) \vee (n \leq Sm). \tag{4.14}$$

We need the following lemmas. □

Lemma 4.21. $m \leq n \vdash (m = n) \vee \exists x \in \mathbf{N}.((m \leq x) \wedge (Sx = n))$.

Proof: We suppose $n = m + k$ and prove the equivalent sequent $\vdash (n = m + k) \Rightarrow ((m = n) \vee \exists x \in \mathbf{N}.((m \leq x) \wedge (Sx = n))$ by formal induction (4.5) on k. The base case clearly holds. For the induction step, it is sufficient to prove $\vdash (n = m + Sk) \Rightarrow ((m = n) \vee \exists x \in \mathbf{N}.((m \leq x) \wedge (Sx = n)))$, where we can replace $m + Sk$ with $S(m + k)$ by Proposition 4.4. Finally, suppose $x \equiv m + k$. □

Lemma 4.22. $\vdash n \leq Sn$.

Proof: Obvious. □

Lemma 4.23. $m \leq n \vdash Sm \leq Sn$.

Proof: In general, we have $n = m + l \vdash Sn = S(m + l)$. By Proposition 4.4, we can replace $S(m + l)$ with $Sm + l$. □

We now turn to prove the linearity. To prove (4.13), we have $m \leq n \vdash (m = n) \vee \exists l \in \mathbf{N}.((m \leq l) \wedge (Sl = n))$ by Lemma 4.21, $m = n \vdash n \leq Sm$ by Lemma 4.22, and $(m \leq l) \wedge (Sl = n) \vdash (Sm \leq Sl) \wedge (Sl = n) \vdash Sm \leq n$ by Lemma 4.23. Applying the cut rule to these sequents yields the desired result. To prove (4.14), we have $\vdash m \leq Sm$ by Lemma 4.22 and $n \leq m, m \leq Sm \vdash n \leq Sm$ by the transitivity of $\leq$. Applying the cut rule to these sequents yields the desired result.

Notation. We write $m < n$ for $\exists x \in \mathbf{N}.((n = m + x) \wedge \neg(x = 0))$.

5. CONSTRUCTING REAL NUMBERS

This section presents a step-by-step description of quantum real numbers. We take natural numbers as basic objects, construct rational numbers from pairs of natural numbers, and characterize real numbers as Dedekind's cuts in the set of rational numbers.

5.1. Rational Numbers

For simiplicity, we continue to restrict ourselves to non-negative numbers. We extend our number objects to include negatives in the final subsection.

We write $\mathbf{N}_+$ for the set of positive natural numbers, i.e, $\mathbf{N}_+ \equiv \{x \in \mathbf{N} | S0 \leq x\}$. We set $\langle m, n\rangle_{\mathbf{Q}} \equiv \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N}_+ | \exists z \in \mathbf{N}_+.(zmy = znx)\}$ for each pair $\langle m, n\rangle$ such that $\vdash \langle m, n\rangle \in \mathbf{N} \times \mathbf{N}_+$. $\langle m, n\rangle_{\mathbf{Q}}$ is called a *quantum rational number*, or simply a *rational*. Intuitively, $\langle m, n\rangle_{\mathbf{Q}}$ corresponds to the usual rational number m/n.

Theorem 5.1. *The $\mathcal{L}^N$-set $\{\langle\langle x, y\rangle, \langle z, w\rangle\rangle \in (\mathbf{N} \times \mathbf{N}_+) \times (\mathbf{N} \times \mathbf{N}_+) | \langle x, y\rangle \in \langle z, w\rangle_Q\}$ is an equivalence relation.*

Proof: We must prove

$$\vdash \langle m, n\rangle \in \langle m, n\rangle_{\mathbf{Q}} \quad \text{(Reflexivity)} \tag{5.1}$$

$$\langle k, l\rangle \in \langle m, n\rangle_{\mathbf{Q}} \vdash \langle m, n\rangle \in \langle k, l\rangle_{\mathbf{Q}} \quad \text{(Symmetry)} \tag{5.2}$$

$$\langle i, j\rangle \in \langle k, l\rangle_{\mathbf{Q}}, \langle k, l\rangle \in \langle m, n\rangle_{\mathbf{Q}} \vdash \langle i, j\rangle \in \langle m, n\rangle_{\mathbf{Q}} \quad \text{(Transitivity).} \tag{5.3}$$

(5.1) and (5.2) clearly hold. The sequent (5.3) means $skj = sli, tml = tnk \vdash x \in \mathbf{N}_+.(xmj = xni)$. Letting $x \equiv lst$ and using the left side equations of the sequent, we obtain the right side equation of the sequent (where we have used the fact that $\mathrm{S}0 \leq m, \mathrm{S}0 \leq n \vdash \mathrm{S}0 \leq mn$). □

The set of all non-negative rationals is represented by the $\mathcal{L}^{\mathbf{N}}$-set $\mathbf{Q} \equiv \{x \in P(\mathbf{N} \times \mathbf{N}) \,|\, \exists y \in \mathbf{N}.(\exists z \in \mathbf{N}_+.(x = \langle y, z\rangle_{\mathbf{Q}}))\}$. We write $n_{\mathbf{Q}}$ for $\langle n, S0\rangle_{\mathbf{Q}}$.

Addition of Q

To define addition of $\mathbf{Q}$, we write $\langle k, l\rangle_{\mathbf{Q}} + \langle m, n\rangle_{\mathbf{Q}}$ for $\langle kn + lm, ln\rangle_{\mathbf{Q}}$. We need to check that $+$ on $\mathbf{Q}$ is a well-defined operation. It is immediate that $\vdash \langle kn + lm, ln\rangle_{\mathbf{Q}} \in \mathbf{Q}$ holds. We must prove $\langle k', l'\rangle \in \langle k, l\rangle_{\mathbf{Q}}, \langle m', n'\rangle \in \langle m, n\rangle_{\mathbf{Q}} \vdash \langle k'n' + l'm', l'n'\rangle \in \langle kn + lm, ln\rangle_{\mathbf{Q}}$, that is, $skl' = slk', tmn' = tnm' \vdash \exists x \in \mathbf{N}_+.(x(kn + lm)(l'n') = x(ln)(k'n' + l'm'))$. This sequent indeed holds for $x \equiv st$.

This $+$, as expected, satisfies the usual properties of addition. In the sequel, we use metavariables $\alpha, \beta, \gamma, \ldots$ to range over quantum rational numbers.

Proposition 5.2. $\vdash 0_{\mathbf{Q}} + \alpha = \alpha$.

Proof: Obvious. □

Proposition 5.3. $\vdash \alpha + \beta = \beta + \alpha$.

Proof: This follows from the properties of $\mathbf{N}$. □

Proposition 5.4. $\vdash (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Proof: This follows from the properties of **N**. □

Lemma 5.5. $m \in \mathbf{N}_+, mn = 0 \vdash n = 0$

Proof: In general, we have $m \in \mathbf{N}_+ \vdash \mathrm{S}0 \le m \vdash \exists x \in \mathbf{N}.(m = S0 + x) \vdash \exists x \in \mathbf{N}.(m = S(0 + x)) \vdash \exists x \in \mathbf{N}.(m = Sx)$. Suppose $m = Sl$. Then we have $m \in \mathbf{N}_+$, $mn = 0 \vdash (Sl)n = 0 \vdash ln + n = 0 \vdash n = 0$ by Proposition 4.12 and 4.9. □

Proposition 5.6. $\alpha + \beta = 0_{\mathbf{Q}} \vdash \alpha = 0_{\mathbf{Q}} \wedge \beta = 0_{\mathbf{Q}}$.

Proof: Suppose $\alpha \equiv \langle k, l\rangle_{\mathbf{Q}}$ and $\beta \equiv \langle m, n\rangle_{\mathbf{Q}}$. Then $\alpha + \beta = 0_{\mathbf{Q}}$ means $\exists x \in N_{+}.(x(kn + lm) = 0)$. We have $kn + lm = 0$ by Lemma 5.5. Hence $kn = 0$ and $lm = 0$ by Proposition 4.9. Using Lemma 5.5 again, we get $k = 0$ and $m = 0$, which means that $\alpha = 0_{\mathbf{Q}}$ and $\beta = 0_{\mathbf{Q}}$. □

Proposition 5.7. $\alpha + \beta = \alpha + \gamma \vdash \beta = \gamma$.

Proof: This follows from the properties of **N**. □

*Multiplication of **Q***

To define multiplication of **Q**, we write $\langle k, l\rangle_{\mathbf{Q}} \cdot \langle m, n\rangle_{\mathbf{Q}}$ for $\langle km, ln\rangle_{\mathbf{Q}}$. We need to check that $\cdot$ is a well-defined operation. It is immediate that $\vdash \langle km, ln\rangle_{\mathbf{Q}} \in \mathbf{Q}$. We must prove $\langle k', l'\rangle \in \langle k, l\rangle_{\mathbf{Q}}, \langle m', n'\rangle \in \langle m, n\rangle_{\mathbf{Q}} \vdash \langle k'm', l'n'\rangle \in \langle km, ln\rangle_{\mathbf{Q}}$, that is, $skl' = slk', tmn' = tnm' \vdash \exists x \in \mathbf{N}_{+}.(x(km)(l'n') = x(ln)(k'm'))$. This sequent indeed holds for $x \equiv st$.

The usual algebraic properties of multiplication such as the existence of identity ($1_{\mathbf{Q}}$), associativity, commutativity, and distributivity are reduced to the properties of the operations of **N**. Besides, it is important that there exists an inverse element for each element in **Q**, except $0_{\mathbf{Q}} : \langle m, n\rangle_{\mathbf{Q}} \cdot \langle n, m\rangle_{\mathbf{Q}} = 1_{\mathbf{Q}}$. We write $\langle m, n\rangle^*_{\mathbf{Q}}$ for $\langle n, m\rangle_{\mathbf{Q}}$.

*5.1.1. Order on **Q***

We set $O_{\mathbf{Q}} \equiv \{\langle x, y\rangle \in \mathbf{Q} \times \mathbf{Q} | \exists z \in \mathbf{Q}.(y = x + z)\}$ and write $\alpha \le \beta$ for $\langle \alpha, \beta\rangle \in O_{\mathbf{Q}}$. To see that this relation $\le$ on **Q** is indeed a linear order relation, we need to check if it satisfies reflexivity, transitivity, antisymmetry and linearity. The first three properties are easily established by using Proposition 5.2–5.4, 5.6, and 5.7. We only show the linearity.

Proposition 5.8. $\vdash (\alpha \le \beta) \vee (\beta \le \alpha)$.

Proof: Suppose $\alpha \equiv \langle \alpha_1, \alpha_2 \rangle_{\mathbf{Q}}$ and $\beta \equiv \langle \beta_1, \beta_2 \rangle_{\mathbf{Q}}$. By the linearity of $O_{\mathbf{N}}$, we have $\vdash (m \leq n) \vee (n \leq m)$ for each natural numbers m, n. Letting $m = \alpha_1\beta_2$ and $n = \alpha_2\beta_1$, we have $\vdash \exists x \in \mathbf{N}.(\alpha_2\beta_1 = \alpha_1\beta_2 + x) \vee \exists y \in \mathbf{N}.(\alpha_1\beta_2 = \alpha_2\beta_1 + y)$. Multiplying $\alpha_2\beta_2$ to both side of the equations yields $\vdash \exists x \in \mathbf{N}.(\alpha_2\beta_2\alpha_2\beta_1 = \alpha_2\beta_2\alpha_1\beta_2 + \alpha_2\beta_2 x) \vee \exists y \in \mathbf{N}.(\alpha_2\beta_2\alpha_1\beta_2 = \alpha_2\beta_2\beta_1\alpha^2 + \alpha_2\beta_2 y)$ Letting $z_1 \equiv x$, $z_2 \equiv \alpha_2\beta_2, w_1 \equiv y$ and $w_2 \equiv \alpha_2\beta_2$, we have $\vdash \exists z_1 \in \mathbf{N}.(\exists z_2 \in \mathbf{N}_{+}.(\beta_1\alpha_1 z_2 = \beta_2\alpha_1 z_2 + \beta_2\alpha_2 z_1)) \vee \exists w_1 \in \mathbf{N}.(\exists w_2 \in \mathbf{N}_{+}.(\alpha_1\beta_2 w_2 = \alpha_2\beta_1 w_2 + \alpha_2\beta_2 w_1))$. Finally, letting $z \equiv \langle z_1, z_2 \rangle_{\mathbf{Q}}$ and $w \equiv \langle w_1, w_2 \rangle_{\mathbf{Q}}$, we obtain $\vdash \exists z \in \mathbf{Q}.(\beta = \alpha + z) \vee \exists w \in \mathbf{Q}.(\alpha = \beta + w)$. □

Notation. We write $\alpha < \beta$ for $\exists x \in \mathbf{Q}.((\beta = \alpha + x) \wedge \neg(x = 0_{\mathbf{Q}}))$.

5.2. Real Numbers and Observables

Real Numbers

The quantum real numbers are constructed from the quantum rationals by *Dedekind cuts*. We say that a term Λ of type $PP(N \times N)$ is a *quantum real number*, or simply a *real*, if it satisfies all the following conditions.

(**R1**) $\Lambda \subseteq \mathbf{Q}$
(**R2**) $\exists x \in \mathbf{Q}.(x \in \Lambda)$
(**R3**) $\exists x \in \mathbf{Q}.\neg(x \in \Lambda)$
(**R4**) $\exists x \in \Lambda.(x < y) \Rightarrow (y \in \Lambda)$
(**R5**) $(x \in \Lambda) \Rightarrow \exists y \in \Lambda.(y < x)$

In other words, setting $\text{Real}(\Lambda) \equiv (\Lambda \subseteq \mathbf{Q}) \wedge \exists x \in \mathbf{Q}.(x \in \Lambda) \wedge \exists x \in \mathbf{Q}.\neg(x \in \Lambda) \wedge (\exists x \in \Lambda.(x < y) \Rightarrow (y \in \Lambda)) \wedge ((x \in \Lambda) \Rightarrow \exists y \in \Lambda.(x < x))$, we consider $\text{Real}(\Lambda)$ as the predicate asserting that "Λ is a quantum real." In the sequel, we use metavariables K, Λ, M, N to range over quantum real numbers. The set of all non-negative reals is represented by the $\mathcal{L}^{\mathbf{N}}$-set $\mathbf{R} \equiv \{x \in P\mathbf{Q} \mid \text{Real}(x)\}$.

We set $\alpha_{\mathbf{R}} \equiv \{x \in \mathbf{Q} \mid \alpha < x\}$ for each quantum rational α. Clearly, $\vdash \alpha_{\mathbf{R}} \in \mathbf{R}$. That is, $\alpha_{\mathbf{R}}$ is a quantum real corresponding to the quantum rational α.

Observables

We now look at quantum reals from the perspectives of ordinary mathematics and physics. The following argument is substantially based on the discussion in Takeuti (1978).

We present here a suggestive interpretation of $\mathcal{L}^N$ along the lines of Section 3. Let D_Ω be the orthomodular lattice of projections on some Hilbert space $\mathbf{H}$, and D_N the set of ordinary natural numbers. 0 is interpreted as 0, and S the usual

$+1$-function. Then clearly, $\mathbf{Q}$ corresponds to the set of ordinary (non-negative) rational numbers, which is denoted by $\mathbf{Q}_O$.

Suppose that Λ is a quantum real. For given ordinary (non-negative) rational α, there exists a corresponding quantum rational, say $\hat{\alpha}$. Using this notation, we set $P_{\hat{\alpha}} \equiv \hat{\alpha} \in \Lambda$. Then the following equations hold.

$$\bigvee_{\alpha \in \mathbf{Q}_O} P_{\hat{\alpha}} = \top \tag{5.4}$$

$$\bigwedge_{\alpha \in \mathbf{Q}_O} P_{\hat{\alpha}} = \bot \tag{5.5}$$

$$P_{\hat{\alpha}} = \bigvee_{\beta < \alpha} P_{\hat{\beta}} \tag{5.6}$$

Let $\mathbf{R}_O$ be the set of ordinary (non-negative) real numbers. For $M, N \in \mathbf{R}_O$ and $E_M \equiv \bigvee_{\alpha < M} P_{\hat{\alpha}}$, the following equations hold.

$$\bigvee_{M \in \mathbf{R}_O} E_M = \top \tag{5.7}$$

$$\bigwedge_{M \in \mathbf{R}_O} E_M = \bot \tag{5.8}$$

$$E_M = \bigvee_{N < M} E_N \tag{5.9}$$

This means that $\{E_M\}_{M \in \mathbf{R}_O}$ can be regarded as a spectral measure on $\mathbf{H}$. In other words, for given quantum real Λ, there exists a corresponding spectral measure $\{E_M\}_{M \in \mathbf{R}_O}$ on $\mathbf{H}$. Using the fact that there is a one-to-one correspondence between spectral measures and self-adjoint operators, we can say that a quantum real is interpreted as an observable on $\mathbf{H}$.

5.3. Full Reals

In this last subsection, we extend the concept of quantum numbers to include negative numbers.

*Addition of **R***

To define addition of $\mathbf{R}$, we write $M + N$ for $\{x \in \mathbf{Q} | (x = \beta + \gamma) \wedge (\beta \in M) \wedge (\gamma \in N)\}$. We need to check that $\vdash M + N \in \mathbf{R}$. It is immediate that $M + N$ satisfies (**R1**)–(**R3**) and (**R5**). To verify (**R4**), we must prove $\vdash ((\alpha \in M + N) \wedge (\alpha < \beta)) \Rightarrow (\beta \in M + N)$. Note that for any rational α except $0_{\mathbf{Q}}$, $(\alpha < \beta) = (\alpha^*\alpha < \alpha^*\beta) = (1_{\mathbf{Q}} < \alpha^*\beta) = (1_{\mathbf{Q}} < \alpha^*\beta)$. holds. Letting $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 \in M$, and $\alpha_2 \in N$, we have $\alpha_1 < \alpha^*\beta\alpha_1$ and $\alpha_2 < \alpha^*\beta\alpha_2$, that is, we have $\alpha^*\beta\alpha_1 \in M$

and $\alpha^*\beta\alpha_2 \in N$. Hence, we obtain $\beta = \alpha^*\beta(\alpha_1 + \alpha_2) = \alpha^*\beta\alpha_1 + \alpha^*\beta\alpha_2 \in M + N$.

This $+$, as expected, satisfies the usual properties of addition.

Proposition 5.9. $\vdash (0_{\mathbf{Q}})_{\mathbf{R}} + \Lambda = \Lambda$.

Proof: It is obvious that $\vdash (0_{\mathbf{Q}})_{\mathbf{R}} + \Lambda \subseteq \Lambda$. We show that $\vdash \Lambda \subseteq (0_{\mathbf{Q}})_{\mathbf{R}} + \Lambda$. Since we have $\vdash (\alpha \in \Lambda) \Rightarrow ((\gamma \in \Lambda) \wedge (\beta \in \mathbf{Q}) \wedge (\alpha = \gamma + \beta) \wedge \neg(\beta = 0_{\mathbf{Q}}))$ by (**R5**) and $\beta \in \mathbf{Q} \wedge \neg (\beta = 0_{\mathbf{Q}})_{\mathbf{R}} \vdash \beta \in (0_{\mathbf{Q}})_{\mathbf{R}}$, we obtain $\vdash (\alpha \in \Lambda) \Rightarrow ((\beta \in (0_{\mathbf{Q}})_{\mathbf{R}}) \wedge (\gamma \in \Lambda) \wedge (\alpha = \beta + \gamma))$, which is the desired result. □

Proposition 5.10. $\vdash M + N = N + M$.

Proof: This follows from the properties of **Q**. □

Proposition 5.11. $\vdash (\Lambda + M) + N = \Lambda + (M + N)$.

Proof: This follows from the properties of **Q**. □

5.3.1. Negative Reals

Our goal is to extend the set of quantum reals **R** to include the additive inverse of each element. We accomplish this by considering equivalence classes as we have done in the case of rationals.

Let $\langle M, N\rangle_{\mathbf{R}} \equiv \{\langle x, y\rangle \in \mathbf{R} \times \mathbf{R} | \exists z \in \mathbf{R}.((M + y) + z = (N + x) + z)\}$ for each pair $\langle M, N\rangle$ such that $\vdash \langle M, N\rangle \in \mathbf{R} \times \mathbf{R}$. Intuitively, $\langle M, N\rangle_{\mathbf{R}}$ corresponds to the usual real number $M - N$.

Theorem 5.12. *The $\mathcal{L}^N$-set $\{\langle\langle x, y\rangle, \langle z, w\rangle\rangle \in (\mathbf{R} \times \mathbf{R}) \times (\mathbf{R} \times \mathbf{R}) \,|\langle x, y\rangle \in \langle z, w\rangle_{\mathbf{R}}\}$. is an equivalence relation.*

Proof: This follows from the properties of **R**. □

Finally, we define addition of the full reals as $\langle K, \Lambda\rangle_{\mathbf{R}} + \langle M, N\rangle_{\mathbf{R}} \equiv \langle K + M, \forall\Lambda + N\rangle_{\mathbf{R}}$. The fact that this $+$ operation is indeed well-defined, commutative and associative is easily verified by the properties of **R**. The zero element is $\langle(0_{\mathbf{Q}})_{\mathbf{R}}, (0_{\mathbf{Q}})_{\mathbf{R}}\rangle_{\mathbf{R}}$, the inverse element of $\langle M, N\rangle_{\mathbf{R}}$ is $\langle N, M\rangle_{\mathbf{R}}$, and any non-negative real Λ naturally corresponds to $\langle\Lambda, (0_{\mathbf{Q}})_{\mathbf{R}}\rangle_{\mathbf{R}}$. Thus, we can say that this representation of numbers by the equivalence classes is regarded as an extension of **N**, **Q**, and **R**.

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